ANALYSIS OF THE REFRACTING LEAKY MODES IN D-SHAPED OPTICAL FIBERS

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Abstract

An analytical solution for the refracting leaky modes of a waveguide with circular core and large circular cladding neighbouring a dielectric halfspace is presented in this article. This model is applied to D-shaped fibers and the results confirm what is verified experimentally: the x-polarization, normal to the surface, irradiates more than the y-polarization, parallel to the surface. The analysis of a structure with the cladding constituted by air is done. Graphics of the behavior of the real and imaginary parts of the effective index as a function of the spatial frequency are obtained for both polarizations as well as the graphics of the power densities, calculated for the fundamental and the first superior modes.

1. Introduction

The theoretical analysis of x and y-polarizations of the bound modes in polished optical fibers has been too much developed in the last decade \cite{1} – \cite{4}. A similar model to the D-shaped fiber was used to model different optical components \cite{5} – \cite{7}. In all these studies, the modes analyzed were the bound modes. Although the leaky modes in optical fibers with W and parabolic index profiles have already been studied and there are several publications about them \cite{1} and \cite{8} – \cite{9}, there aren’t too many publications about the refracting leaky modes in D-shaped fibers.

The leaky modes are modes that propagate in the guide irradiating power. They appear in continuity of the dispersion curves of bound modes in the region below the cut off frequency. In classical optical fibers, two types of leaky modes are known: the tunneling leaky modes, that arise because of the curvature of the boundary between the core and the cladding and the refracting leaky modes, that arise from the beams that fall upon in the boundary with angles smaller than the critical angle \cite{1}. Since the surface of the D-shaped fiber is plane, the leaky modes considered in this article are just the refracting ones.

The modeling of the refractive leaky modes developed here is similar to that one that is used in the bound modes \cite{2} and \cite{4} and the geometry of the guide studied is that shown in Fig. 2.

The two models differ in the propagation constant: while the propagation constant of the leaky modes considerate the irradiated power and, therefore, is a complex number, the propagation constant of the bound modes is a straight imaginary number because there is no irradiated energy, considering only the term related to the wave’s phase.

The methodology adopted includes two stages. First, the cut off frequencies of the different propagating modes in the D-shaped fiber are calculated. Using these values and the refracting leaky modes eigenvalues equation, the curves related to the real and imaginary parts of the effective index as a function of the spatial frequencies smaller than the respective cut off frequencies are obtained.
For both polarizations, the curves of the real and imaginary parts of the refractive index as a function of the spatial frequency for different values of the normalized distance D of the plane to the core and the graphics of the power densities for the fundamental and the first superior modes are obtained.

2. Mathematical Model

An implicit time dependence $e^{j\omega t}$ was assumed. The waveguide is uniform and only the forward propagation waves are considerate. The longitudinal component equations of the electric and magnetic fields are uncoupled in the regions of constant index.

The mathematical model adopted follows the block diagram shown in Fig.1.

Thus, at first, the following pair of equations must be solved at each region of the structure:

$$\left[ \nabla^2 + \left( k_0^2 \eta^2 - \beta^2 \right) \right] e_z = 0$$
$$\left[ \nabla^2 + \left( k_0^2 \eta^2 - \beta^2 \right) \right] h_z = 0$$

The transversal components are obtained by the hybrid formulation [1].

A. Fields in the core and in the cladding

In the core, the y-polarization state (quasi-TE mode) consists of the expansion of the normal modes expressed in polar coordinate system as follow:

$$e_z = \sum_{n=0}^{\infty} A_n J_n(UR) \text{sen}(n\theta)$$
$$h_z = \sum_{n=0}^{\infty} B_n J_n(UR) \cos(n\theta)$$

In the cladding, the corresponding solution is given by:
\[ e_z = \sum_{n=0}^{\infty} \left[ C_n K_n(\text{WR}) + D_n I_n(\text{WR}) \right] \text{sen}(n\theta) \]
\[ h_z = \sum_{n=0}^{\infty} \left[ E_n K_n(\text{WR}) + F_n I_n(\text{WR}) \right] \text{cos}(n\theta) \]

where \( A_n, B_n, C_n, D_n, E_n \) and \( F_n \) are constants to be determined, \( J_n, K_n \) and \( I_n \) are Bessel functions of order \( n \), \( R \) is the normalized radial coordinate \((R = r/a)\) and \( U, W \) are the normalized modal parameters, which are defined as

\[ U = a\sqrt{k^2_i - \beta^2} \]
\[ W = a\sqrt{\beta^2 - k^2_i} = \sqrt{V^2 - U^2} \]
\[ \beta = k_0(\eta_{d} - j \eta_{d}) \]  

In these equations, \( V \) is the normalized spatial frequency.

The x-polarization mode (quasi-TM mode) is obtained interchanging the sine and cosine functions in (2) and (3).

**B. Planar representation of the cladding fields**

Since the fields in the halfspace are expanded in terms of continuous plane waves, the representation of the cladding fields by Fourier-Bessel expansions are not appropriate. So, we use the same approach of [2] to overcome this difficulty.

The cladding fields in the neighborhood of the plane interface are expanded in the rectangular system by plane waves. These expressions are valid for both polarizations.

\[ e_z = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ P_v e^{-\sigma(X-D)} + Q_v e^{\sigma(X-D)} \right] e^{-\beta Y} d\nu \]
\[ h_z = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ R_v e^{-\sigma(X-D)} + S_v e^{\sigma(X-D)} \right] e^{-\beta Y} d\nu \]

where \( P_v, Q_v, R_v \) and \( S_v \) are constants to be determined, \( X = x/a \) and \( Y = y/a \) are the normalized coordinates, \( D \) is the normalized distance between the half plane and the center of the fiber core \((D = d/a)\) and \( a \) is the radius of the core.

In order to simplify the final expressions of the model, we define the \( g \) variable as

\[ \cosh(g) = \frac{\sigma}{W} \]  

Considering the characteristics of the propagating wave in the \( y \) direction and evanescent wave in \( x \) direction, the separation equation for the cladding can be written as

\[ \sigma^2 = (ak_0)^2 \left( \eta_{d}^2 - \eta_{2}^2 \right) + \nu^2; \quad \eta_{d} = \eta_{d} - j \eta_{d} \]

where \( \sigma \) and \( \nu \) are complex numbers.

Using the equations (6) and (7) and the value of \( W \) in (4), we obtain the value of \( \nu \) as a function of the \( g \) variable, that is

\[ \nu = W \text{senh}(g) \]
The constants of (5) are related to (3) by the following expressions:

\[
D_s = j2 \int_{0}^{\infty} Q_s e^{-\sigma \lambda} \sinh(n \lambda) d\nu
\]

\[
F_s = j \frac{2}{\varepsilon_s} \int_{0}^{\infty} S_s e^{-\sigma \lambda} \cosh(n \lambda) d\nu
\]

\[
P_v = \frac{e^{-\sigma \lambda}}{2\sigma} \sum_{n=0}^{\infty} C_n \sinh(n \lambda)
\]

\[
R_v = \frac{e^{-\sigma \lambda}}{2\sigma} \sum_{n=0}^{\infty} E_n \cosh(n \lambda) d\nu
\]

(9)

for y-polarization and

\[
D_s = j2 \int_{0}^{\infty} Q_s e^{-\sigma \lambda} \sinh(n \lambda) d\nu
\]

\[
F_s = j \frac{2}{\varepsilon_s} \int_{0}^{\infty} S_s e^{-\sigma \lambda} \cosh(n \lambda) d\nu
\]

\[
P_v = \frac{e^{-\sigma \lambda}}{2\sigma} \sum_{n=0}^{\infty} C_n \sinh(n \lambda)
\]

\[
R_v = \frac{e^{-\sigma \lambda}}{2\sigma} \sum_{n=0}^{\infty} E_n \cosh(n \lambda) d\nu
\]

(10)

for x-polarization, where \(\varepsilon_n = 2\) if \(n = 0\) and 1 otherwise.

C. Fields in the halfspace

These expressions are valid for both polarizations:

\[
e_v = \int_{0}^{\infty} G_v e^{-j\nu \lambda} e^{-j(X-D)} d\nu
\]

\[
h_v = \int_{0}^{\infty} H_v e^{-j\nu \lambda} e^{-j(X-D)} d\nu
\]

(11)

where \(G_v\) and \(H_v\) are functions of the variable \(\nu\) to be determined.

The \(\{e^{-j\nu \lambda}, e^{-j(X-D)}\}\) terms confirm that the plane waves are irradiated from the fiber, while the energy propagates in the core.

The field components in the halfspace are obtained by the hybrid equations [1] – [2].

D. Pseudo-scattering matrix

The coefficients of (5) are related to the coefficients of (11) through the boundary conditions in the plane surface \(X = D\). This results in an matrix called pseudo-scattering matrix.

\[
\begin{bmatrix}
Q_v \\
S_v
\end{bmatrix} = \begin{bmatrix}
A_{11}(\nu) & jA_{12}(\nu) \\
-jA_{21}(\nu) & A_{22}(\nu)
\end{bmatrix} \begin{bmatrix}
P_v \\
R_v
\end{bmatrix}
\]

(12)
The name pseudo is used to emphasize that this matrix is not really the scattering one because the normalizing conditions are not verified; this matrix only relates the electric and magnetic reflected waves in the boundary with their respective incident waves. The parameters of the matrix are:

\[
A_{ii}(\nu) = \frac{1}{\gamma} \left[ \sigma(\eta_{Q_i}^i - j\tauQ_i^i) \right] (\sigma Q_i^i - j\tau Q_i^i) + (\eta_{\nu}^i \nu V_i^i)^i
\]

\[
A_{ij}(\nu) = \frac{1}{\gamma} \left[ 2Z_i \eta_{\nu}^i \nu \sigma(\eta_{Q_i}^i) \right]
\]

\[
A_{ii}(\nu) = \frac{1}{\gamma} \left[ 2 \left( \frac{\eta_{\nu}^i}{Z_i} \right) \nu \sigma(\eta_{Q_i}^i) \right]
\]

\[
A_{ij}(\nu) = \frac{1}{\gamma} \left[ \sigma(Q_i^i - j\tau Q_i^i) \right](\sigma Q_i^i + j\tau Q_i^i) + (\eta_{\nu}^i \nu V_i^i)^i
\]

\[
\gamma = [\sigma(\eta_{Q_i}^i) - j\tau(\eta_{Q_i}^i)](\sigma Q_i^i - j\tau Q_i^i) - (\eta_{\nu}^i \nu V_i^i)^i
\]

\[
Q_i^2 = \frac{V^2(\eta_{\nu}^2 - \eta_{\nu}^2)}{(\eta_{\nu}^2 - \eta_{\nu}^2)}
\]

\[
Q_i^2 = \frac{V^2(\eta_{\nu}^2 - \eta_{\nu}^2)}{(\eta_{\nu}^2 - \eta_{\nu}^2)}
\]

\[
V_i^2 = Q_i^2 + Q_i^2 = \frac{V^2(\eta_{\nu}^2 - \eta_{\nu}^2)}{(\eta_{\nu}^2 - \eta_{\nu}^2)}
\]

\[
Z_o = 120\pi
\]

Replacing (8) into (13), the pseudo-scattering matrix is expressed in the $g$-plane.

3. Eigenvalues Matrix

In this section, the eigenvalues matrix equations for leaky modes are found for both polarizations.

The boundary conditions between the core and the cladding allows to express the cladding’s coefficients by the core’s ones.

The eigenvalues matrix equation for $y$-polarization is

\[
\begin{bmatrix}
C_n \\
D_n \\
E_n \\
F_n
\end{bmatrix} =
\begin{bmatrix}
(J_iI_n + \alpha J_iI_n) & \left( -n\alpha J_iI_n \right) \\
\left( J_nK_n + \alpha J_nK_n \right) & \left( -n\alpha J_nK_n \right) \\
\left( -n\alpha J_iI_n \right) & \left( J_iI_n + \alpha J_iI_n \right) \\
\left( -n\alpha J_nK_n \right) & \left( J_nK_n + \alpha J_nK_n \right)
\end{bmatrix}^A
\]

\[
\begin{bmatrix}
A_n \\
B_n
\end{bmatrix}
\]

while the eigenvalues matrix equation for $x$-polarization is

\[
\begin{bmatrix}
C_n \\
D_n \\
E_n \\
F_n
\end{bmatrix} =
\begin{bmatrix}
(J_iI_n + \alpha J_iI_n) & \left( n\alpha J_iI_n \right) \\
\left( J_nK_n + \alpha J_nK_n \right) & \left( n\alpha J_nK_n \right) \\
\left( n\alpha J_iI_n \right) & \left( J_iI_n + \alpha J_iI_n \right) \\
\left( n\alpha J_nK_n \right) & \left( J_nK_n + \alpha J_nK_n \right)
\end{bmatrix}^A
\]

\[
\begin{bmatrix}
A_n \\
B_n
\end{bmatrix}
\]

where:
The coefficients \((Q_\nu, S_\nu, P_\nu, R_\nu)\) are expressed as functions of \((C_n, D_n, E_n, F_n)\) with the help of (9) and (12) for y-polarization and (10) and (12) for x-polarization.

The eigenvalues matrix equation of the leaky modes for y-polarization is found by (14):

\[
\begin{bmatrix}
K_1 & K_2 \\
K_3 & K_4
\end{bmatrix}
\begin{bmatrix}
B_n \\
A_n
\end{bmatrix} = 0
\]

\[
K_1 = \varepsilon_y T_{2n} + \sum_{m=0}^{\infty} \left[ T_{4mn} S_{11}^{mn} - T_{3mn} S_{12}^{mn} \right]
\]
\[
K_2 = \varepsilon_y T_{3n} + \sum_{m=0}^{\infty} \left[ T_{4mn} S_{11}^{mn} - T_{3mn} S_{12}^{mn} \right]
\]
\[
K_3 = T_{5n} + \sum_{m=0}^{\infty} \left[ T_{4mn} S_{21}^{mn} - T_{5mn} S_{22}^{mn} \right]
\]
\[
K_4 = T_{4n} + \sum_{m=0}^{\infty} \left[ T_{5mn} S_{11}^{mn} - T_{4mn} S_{12}^{mn} \right]
\] (17)

and, for x-polarization, by (15):

\[
\begin{bmatrix}
K_1 & K_2 \\
K_3 & K_4
\end{bmatrix}
\begin{bmatrix}
A_n \\
B_n
\end{bmatrix} = 0
\]

\[
K_1 = \varepsilon_x T_{2n} + \sum_{m=0}^{\infty} \left[ T_{3mn} S_{11}^{mn} - T_{4mn} S_{12}^{mn} \right]
\]
\[
K_2 = \varepsilon_x T_{3n} + \sum_{m=0}^{\infty} \left[ T_{3mn} S_{11}^{mn} - T_{4mn} S_{12}^{mn} \right]
\]
\[
K_3 = T_{5n} + \sum_{m=0}^{\infty} \left[ T_{3mn} S_{21}^{mn} - T_{5mn} S_{22}^{mn} \right]
\]
\[
K_4 = T_{4n} + \sum_{m=0}^{\infty} \left[ T_{5mn} S_{11}^{mn} - T_{4mn} S_{12}^{mn} \right]
\] (18)

where:

\[
T_{2n} = (J_1 I_n + \alpha_1 J_1 n) \quad T_{3n} = n \alpha_1 J_1 n
\]
\[
T_{3n} = (J_2 K_n + \alpha_2 J_2 n) \quad T_{5n} = n \alpha_2 J_2 n
\]
\[
T_{4n} = (J_1 I_n + \alpha_3 J_1 n) \quad T_{6n} = n \alpha_3 J_1 n
\]
\[
T_{5n} = (J_2 K_n + \alpha_4 J_2 n) \quad T_{6n} = n \alpha_4 J_2 n
\] (19)

The integral terms for y-polarization are:
\[ S_{m11}^{\text{mm}} = \int_{g=-\infty}^{g=\infty} A_{22}(g) \cosh(ng) \cosh(mg) e^{-2\rho D} \, dg \]
\[ S_{m12}^{\text{mm}} = \int_{g=-\infty}^{g=\infty} A_{21}(g) \cosh(ng) \sinh(mg) e^{-2\rho D} \, dg \]
\[ S_{m21}^{\text{mm}} = \int_{g=-\infty}^{g=\infty} A_{12}(g) \sinh(ng) \cosh(mg) e^{-2\rho D} \, dg \]
\[ S_{m22}^{\text{mm}} = \int_{g=-\infty}^{g=\infty} A_{11}(g) \sinh(ng) \sinh(mg) e^{-2\rho D} \, dg \]  

(20)

and the integral terms for x-polarization are:

\[ S_{x11}^{\text{mm}} = \int_{g=-\infty}^{g=\infty} A_{11}(g) \cosh(ng) \cosh(mg) e^{-2\rho D} \, dg \]
\[ S_{x12}^{\text{mm}} = \int_{g=-\infty}^{g=\infty} A_{12}(g) \cosh(ng) \sinh(mg) e^{-2\rho D} \, dg \]
\[ S_{x21}^{\text{mm}} = \int_{g=-\infty}^{g=\infty} A_{21}(g) \sinh(ng) \cosh(mg) e^{-2\rho D} \, dg \]
\[ S_{x22}^{\text{mm}} = \int_{g=-\infty}^{g=\infty} A_{22}(g) \sinh(ng) \sinh(mg) e^{-2\rho D} \, dg \]  

(21)

4. Results

In this section, several results obtained by the theory developed in this work are presented.

The waveguide geometry of the analysed polished optical fiber is shown at Fig. 2.

![Waveguide geometry](image)

Fig. 2 – Waveguide geometry

The cross section shows the radius \( a \) of the circular core and \( d \) is the halfspace distance from the core. The refractive indexes are: \( \eta_1 = 1.447 \), \( \eta_2 = 1.00 \) and \( \eta_3 = 1.40 \).
Fig. 3 shows the behavior of the imaginary part of the effective index ($\eta_{\text{eff}}$) as a function of the normalized frequency $V$ for the y-polarization.

This term is responsible for the irradiated power of the structure (leaky power). The cut off frequency of the analysed mode is 6.638. In Fig. 3, we note that the irradiation increases due to two factors: the decrease of the frequency and the asymmetry degree of the structure (decreasing $D$, the structure becomes more asymmetric).

In Fig. 4, we studied the behavior of ($\eta_{\text{eff}}$) for the x-polarization.

Fig. 4 – Imaginary part of the effective index versus the normalized frequency for the x-polarization
Comparing Fig. 3 and Fig. 4, we note that the x-polarization irradiates, on average, 4 times more than the y-polarization. This results can explain the excitement of “plasmon” (conductor layer electrons placed over the plane surface of D-shaped fibers) that is made by the x-polarization (quasi-TM mode). From the analysis done in the theory developed here, the x-polarization irradiates more than the y-polarization, what is in agreement with the referenced experiment.

Fig. 5 shows the behavior of the real part of the effective index \( \eta_{ref} \) as a function of the normalized frequency for the y-polarization.

![Fig. 5 – Real part of the effective index versus the normalized frequency for the y-polarization](image)

We note that the value of \( \eta_{ref} \) decreases as we move away from the cut off frequency. We also note that as D decreases, the guide becomes more dense and the real part of the effective index \( \eta_{ref} \) increases.

Fig. 6 shows the behavior of \( \eta_{ref} \) as a function of V for the x-polarization.
For this polarization, the \((\eta_{ref})\) parameter is weakly influenciated by the asymmetry of the structure, but the same characteristics noted in y-polarization are also present in this polarization.

Next, are shown the graphics of the power densities for both polarizations calculated for the fundamental mode \((V = 4.829)\) and the first superior mode \((V = 6.321)\).

All these graphics are related to the Fig. 7, that shows a transversal section of the analyzed structure. This was done to facilitate the comprehension of the graphics about the power density distributions.

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Fig. 6 – Real part of the effective index \(\eta_{ref}\) versus the normalized frequency for the x-polarization

Fig. 7 - Planar view of the structure

Fig. 8 shows the power density for the y-polarization of the fundamental mode.
In this polarization, the maximum distribution of the energy is dislocated from the vertical axis of the structure.

Fig. 9 shows the power density for the x-polarization of the fundamental mode.

In this case, the maximum distribution of the energy is concentrated in the vertical axis of the guide.

Fig. 10 shows the power density for the y-polarization of the first superior mode.
Comparing to the fundamental mode, the energy is accentuated far from the vertical axis of the fiber.

Fig. 11 shows the power density for the x-polarization of the first superior mode.

In this case, two peaks of energy appear, resulting in a mode with a greater number of variations in the electric field.
5. Conclusion

The results supplied by the theory developed in this work about the leaky modes in D-shaped optical fibers are in agreement with the practical results; they explain why the plasmon is excited by the x-polarization and confirm the properties expected for a waveguide that, when the dielectric section increases, the real part of the effective index increases and also when the asymmetry of the structure increases, the irradiated power increases, increasing the imaginary part of the effective index.

The graphics about the power density distributions show that the convenient polarization to use the irradiated energy is the x-polarization, because in the fundamental mode of this polarization the energy is concentrated in a single direction, the vertical axis of the structure.

Those results must be confirmed by other methods or practical measurements, what will be made in a near future.

The use of this method demands a great deal of care and practice. The convergence of the method was obtained using twenty Bessel functions with a $40 \times 40$ eigenvalue matrix similar to (17) and (18).

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